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Ideals in Group algebra of Heisenberg Group

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Abstract— In spectral theory ideals are very important. We derive the relation between non commutative and commutative algebra by a transformation which is associated to the semi-direct product of groups. We obtain and classify the ideal in L^1-algebra of Heisenberg group.

Index Terms— Heisenberg group, Ideals in L^1-algebra of the Heisenberg group, Semi-direct product.

1 Introduction

We recall some definitions.

Definition 1.1: The *Heisenberg group* is the group of 3 X 3 upper triangular matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$

Definition 1.2: For $a \in \mathbb{R}^n, b \in \mathbb{R}^n, c \in \mathbb{R}$ and I_n (Identity matrix of order n), the *Heisenberg group of dimension* $2n+1$ is the group of upper triangular matrices of the form

$$\begin{pmatrix} 1 & a & b \\ 0 & I_n & c \\ 0 & 0 & 1 \end{pmatrix} \quad (1.1)$$

Let $\mathrm{Aut}(\mathbb{R}^{n+1})$ is the group of all automorphisms of $\mathbb{R}^{n+1}$ then for any $a=(a_1,a_2,...a_n)\in\mathbb{R}^n$ $b=(b_1,b_2,...b_n)\in\mathbb{R}^n$, $c\in\mathbb{R}$ and $ab=\sum_{i=1}^{n} a_i b_i$, define $G=\mathbb{R}^{n+1} \rtimes_\rho \mathbb{R}^n$ be the group of semi direct product of $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$ by the group homomorphism $\rho:\mathbb{R}^n \to \mathrm{Aut}(\mathbb{R}^{n+1})$ which is defined by,

ρ (a) (b,c)= (b + ac , c) (1.2)

The inverse of an element in G is defined by

For X= ((a, b); c) $\in$ G ,

X^{-1} = ((a, b); c)$^{-1}$
= (−c(−a,−b); −c)
= ((−a + bc,−b); −c) (1.3)

Where −c(−a,−b) = ρ (−c) (−a,−b)

The multiplication of two elements X and Y in G is defined by

For X = ((a, b); c) , Y = ((a’, b’); c’) $\in$ G ,

X · Y = ((a, b); c) ((a’, b’); c’)
= ((a, b) + c(a’, b’); c + c’)
= ((a, b) + (a’+b’c ,b’); c + c’)
= ((a+a’+b’c , b+b’); c + c’) (1.4)

Where c (a’,b’) = ρ (c) (a’,b’)

By mean of group isomorphism $\Psi: G \to H^n$ defined by $\Psi((c,b);a) = \begin{pmatrix} 1 & a & b \\ 0 & I_n & c \\ 0 & 0 & 1 \end{pmatrix}$ the group H^n with the group G can be identified.

Definition 1.3: Let L^1(M) be the Banach algebra that consists of all complex valued functions on the group M - an unimodular Lie group, which are integrable with respect to the Harr measure of M and multiplication is defined by convolution on M.

Let us denote the restriction of L^1(M) on any subgroup N of M by $L^1(M)|_N$. Then $L^1(M)|_N = \{F|_N : F \in L^1(M)\}$ where $F|_N$ is the restriction of the function F on N.

2 Priliminaries.

Let J is real vector group which is direct product of $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$ and K is real vector group which is direct product of G and $\mathbb{R}^n$ then the group G can be identified with the closed subgroup G x {0} of K and J can be identified with the closed subgroup $\mathbb{R}^{n+1} \rtimes_\rho$ {0}x$\mathbb{R}^n$ of K.

Let L = $\mathbb{R}^{n+1}$ x $\mathbb{R}^n$ x $\mathbb{R}^n$ be the group of the direct product of $\mathbb{R}^{n+1}$, $\mathbb{R}^n$ and $\mathbb{R}^n$.

The inverse of element X in L is defined by

For X = ((a, b); c,d) $\in$ L ,

X^{-1} = ((a, b); c, d)$^{-1}$
= (−d (−a,−b); −c, −d)
= (−a+bd, −b); −c, −d) (2.1)

Where −d (−a,−b)= ρ (−d) (−a,−b)

The multiplication of two elements X and Y in L is defined by

For X = ((a, b); c,d) , Y = ((a', b'); c',d') ∈ L ,

X · Y = ((a, b); c ,d) ((a', b'); c',d')
= ((a, b)+d(a', b'); c+c' ,d+d')
= ((a, b)+(a'+b'd, b') ; c+c' ,d+d')
= ((a+a'+b'd, b+b') ; c+c' ,d+d') (2.2)

Where d(a', b') = ρ (d)(a', b').

In such a case the group G can be identified with the closed subgroup $\mathbb{R}^{n+1} \rtimes_\rho \{0\} \times \mathbb{R}^n$ of L and J can be identified with the closed subgroup $\mathbb{R}^{n+1} \times \{0\} \rtimes_\rho \mathbb{R}^n$ of L.

Let the subspace of all complex valued functions on L is denoted by L^1_E(L) such that $L^1_E(L)|_G = L^1(G)$ and $L^1_E(L)|_J = L^1(J)$.

Definition 2.1 For every $f \in L^1_E$(L) , define a function f^* as follows. For all ((a, b); c, d) ∈ L ,

f^* ((a, b); c, d) = f (c(a, b); 0, d + c) (2.3)

It is noted that for all ((a, b); c, d) ∈ L and k ∈ $\mathbb{R}^n$ the function f^* is invariant because,

f^* (k(a,b); c – k, d + k) = f^* ((a, b); c, d) (2.4)

Further it should be noted that restricted functions $f^*|_G \in L^1(G)$ and $f^*|_J \in L^1(J)$.

Definition 2.2: For every v ∈ L^1 (G) or v ∈ L^1 (J) and for any F ∈ L^1_E (L) two convolutions product on the group L are defined by,

(i) v ∗ F ((a,b) ; c, d)

$$= \int_G F[((x, y); z)^{-1}((a,b); c, d)]u((x, y); z)dxdydz$$

$$= \int_G F[-z(a-x, b-y); c, d-z)]u((x, y); z)dxdydz \quad (2.5)$$

(ii) v ∗$_c$ F ((a, b) ; c, d)

$$= \int_J F[(a-x, b-y); c-z, d)]u((x, y); z)dxdydz \quad (2.6)$$

where $dx\ dy\ dz$ is the lebesgue measure on group G.

Corollary 2.1: For each v ∈ (G) , F ∈ L^1_E (L) and for all ((a, b); c, d) ∈ L

v ∗ F((a, b); c, d) = v ∗$_c$ F((a, b); c, d)

proof of this lemma is easily given by the help of (2.4), (2.5), (2.6).

Corollary 2.2 The mapping $\Pi : L^1(G) \to L^1(G)$ defined by

$$\Pi(f^*|_G)((a, b), 0, c) = f^*|_G(c(a, b), 0, c)$$

is a topological isomorphism.

Proof follows from the fact that mapping Π is continuous and its inverse Π^{-1} defined by, $\Pi^{-1}(f^*|_G)((a, b), 0, c) = f^*|_G((-c(a, b)), 0, c)$ is also continuous.

Corollary 2.3 The mapping $\Omega : L^1(J) \to L^1(G)$ defined by

$$\Omega(f^*|_J)((a, b), 0, c) = f^*|_G(c(a, b), 0, c)$$

is a topological isomorphism.

Proof follows from the fact that mapping Ω is continuous and its inverse Ω^{-1} defined by, $\Omega^{-1}(f^*|_G)((a, b), c, 0) = f^*|_J((-c(a, b)), c, 0)$ is also continuous.

Remark: For $I \in L^1_E(\mathrm{L}), I^*$ is the image of I under mapping ∗. Let E = $I^*|_G$, then $I|_G = I^*|_G$ = E

Theorem 1. *Let I be a subset of* $L^1_E(\mathrm{L})$, *then* $\mathrm{E} = \mathrm{I}|_G$ *is a left ideal in the algebra* $L^1(J)$ *if and only if* $I' = I|_J$ *is an ideal in the algebra* $L^1(J)$.

Proof: First suppose $I' = I|_J$ is an ideal in the algebra $L^1(J)$.

By considering the group $K = G \times \mathbb{R}^n$ and the mapping $f \to f^*$ which is defined by,

$$f^*((a, b); c, d) = f\ (-c(a, b); 0, c+d)$$

it is easy to show that $I' = I|_J$ is an ideal in the algebra $L^1(J)$.

Conversely suppose that $I' = I|_J$ is an ideal in the algebra $L^1(J)$.

We know that $v *_c I' \subseteq I'$ and $v *_c I \subseteq I$ for any $v \in L^1(J)$ where

$v *_c I' = \{v *_c (F|_J), F \in I\}$ and

$v *_c I = \{v *_c F, F \in I\}$.

If I^* is the image of I under the mapping ∗, then we have, $v * I^* = v *_c I^* \subseteq I$

and by taking the restriction on the group *G*,

we have $v * I^*|_G = v * I|_G \subseteq I|_G$

So that $v * E \subseteq E$.

This proves that $\mathrm{E} = \mathrm{I}|_G$ is a left ideal in the algebra

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$L^1(J)$.

3. Results

From the above theorem the following results can be verified easily.

(i) If Γ be a subspace of the space $L^1(K)$ such that $I = \Gamma^* |_J$ is an ideal in $L^1(J)$, then (*i*) $I = \Gamma^* |_J$ is a maximal ideal in the algebra $L^1(J)$ if and only if $E = \Gamma |_G$ is a left maximal ideal in the algebra $L^1(G)$.

(*ii*) $I = \Gamma^* |_J$ is a closed ideal in the algebra $L^1(J)$ if and only if $E = \Gamma |_G$ is a left closed ideal in the algebra $L^1(G)$.

(*iii*) $I = \Gamma^* |_J$ is a dense ideal in the algebra $L^1(J)$ if and only if $E = \Gamma |_G$ is a left dense ideal in the algebra $L^1(G)$.

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